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Associated primes of local cohomology and S_2-ification☆

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ABSTRACT

Let R be commutative Noetherian, $I \subset R$ an ideal, and M a finitely generated R-module. We prove that if R/P has an S_2-ification for all $P \in \operatorname{Spec}(R)$ then the set of primes associated to the second local cohomology module $H_I^2(M)$ is finite when $\operatorname{ht}(IR/P) \geq 2$ for all $P \in \operatorname{Ass}_R M$ and $\operatorname{Ass}_R M \subseteq \operatorname{Ass}_R R$. We use that to show that if $\dim(R) = 3$ and the ideal transform of R with respect to any height 2 ideal generated by non-zerodivisors is a finitely generated module, then $\operatorname{Ass}_R H_I^i(M)$ is finite for any I with $\operatorname{ht}(IR/P) \geq 2$. We also reduce the problem of showing $\operatorname{Ass}_R H_I^i(M)$ is finite for local four dimensional rings to an extremely concrete case.

1. Introduction

Let R be a commutative Noetherian ring with identity, $I \subset R$ an ideal, and M a finitely generated R-module. An interesting question is when the ith local cohomology module, $H_I^i(M) := \varinjlim \operatorname{Ext}_R^i(R/I^n, M)$, of M with respect to I has finitely many associated primes. Work on this question in the case where $M = R$ and the ring is regular has been done many people including Lyubeznik, Sharp, Huneke and many others, but in this paper we will look at the case where the dimension of the ring is small while taking local cohomology of a general module. Work in this direction has been done by Hellus, Marley, Huneke and many others. Marley has shown [4] that if R is local of dimension at most three then $\operatorname{Ass}_R H_I^i(M)$ is always finite. He also gets the same result for local rings of dimensions 4 and 5 under various additional conditions.

However, such results do not hold in rings of arbitrarily large dimension as shown by examples in papers of Katzman, Singh, and Swanson. In [6], Singh and Swanson give an examples of a five-dimensional local ring and a four-dimensional non-local ring and an ideal of height 1 so that there are infinitely many primes associated to $H_I^2(R)$.

In this paper we explore the open cases between the theorems of Marley and the counterexamples of Singh and Swanson. In Section 2.1 we extend Marley's local three-dimensional results to the non-local case if the ideal has height at least 2 on M. Specifically, we prove that if a three-dimensional ring has finitely generated ideal transform $D_{(x,y)}(R)$ whenever x and y are non-zerodivisors with $\operatorname{ht}(x, y) = 2$ then $\operatorname{Ass} H_I^i(M)$ is finite for any ideal with $\operatorname{ht}(IR/P) \geq 2$ for every associated prime, P, of M. Because Singh and Swanson's counterexamples all use an ideal of height 1, there is hope that if the height of the ideal is greater than 1 the local cohomology with respect to that ideal could still always have finitely many associated primes even in larger dimensional rings. Since our results here are proved by mapping M to a module which is S_2, it would be interesting to investigate when it is possible to map to modules which are S_n for $n \geq 3$ to handle the case of ideals with larger heights.

If the height of the ideal is 1, there are two open cases: four-dimensional local rings, and three-dimensional rings that are not local. In Section 2.2 we reduce showing that $\operatorname{Ass}_R H_I^i(M)$ is always finite for a local four-dimensional ring and an ideal of height 1 to an extremely concrete special case and discuss how much of that reduction can be done if the ring is three-dimensional and not local.

For a basic introduction to the theory of local cohomology see [5,1]. Below we list some of the basic properties of local cohomology modules and their associated primes used in the next sections.

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Proposition 1. *$H^i_I(M) = 0$ if $i < depth_I M$ or $i > dim(R)$.*

Proposition 2 ([4, Proposition 1.1]). *(a) $Ass_R H^i_I(M) = Ass_R Hom_R(R/I, H^i_I(M))$.*
(b) $Ass_R H^g_I(M) = Ass_R Ext^g_R(R/I, M)$ where $g = depth_I M$; thus $Ass_R H^g_I(M)$ is finite.
(c) $Ass_R H^i_I(M)$ is finite if $i = 0, 1$.

The proofs of [4, Proposition 1.1 (a)–(c)] hold in the case where R is not local.

Proposition 3 (Corollary 2.4 [4]). *If $dim(R) = d$ then $Ass_R H^d_I(M)$ is finite for any ideal $I \subset R$ and finitely generated module M.*

For an introduction to ideal transforms see [2] or [3].

2. Main results

2.1. Ideals of height 2

In this section, we consider the case where the ideal I has height at least two on our module M, i.e., the height of IR/P is at least two for every associated prime P of M. Our goal will be to come close to embedding M into a module on which I has depth at least two, since this will guarantee $H^2_I(M)$ has only finitely many associated primes.

Proposition 4. *Let R be a Noetherian ring, $I \subset R$ an ideal, and M a finitely generated R-module. If there is a map of R-modules $\theta : M \to N$, where N is finitely generated and $depth_I N \geq 2$ and $dim(ker(\theta)) \leq 1$, then $Ass_R H^2_I(M)$ is finite.*

Proof. Let $K = \ker(\theta)$, $M' = M/K$ and $C = N/M'$. The short exact sequence $0 \to M' \to N \to C \to 0$, induces a long exact sequence

$$\cdots \to H^1_I(N) \to H^1_I(C) \to H^2_I(M') \to H^2_I(N) \to \cdots$$

Since $\mathrm{depth}_I N \geq 2$ we know $H^1_I(N) = 0$, so this sequence becomes

$$0 \to H^1_I(C) \to H^2_I(M') \to H^2_I(N) \to \cdots$$

By Proposition 2(b) and (c) we know $\mathrm{Ass}_R H^1_I(C)$ and $\mathrm{Ass}_R H^2_I(N)$ are finite, so $\mathrm{Ass}_R H^2_I(M')$ is finite as well.

The short exact sequence $0 \to K \to M \to M' \to 0$ induces

$$\cdots \to H^2_I(K) \to H^2_I(M) \to H^2_I(M') \to \cdots$$

but $\dim(K) < 2$, so $H^2_I(K) = 0$. Thus $H^2_I(M) \subseteq H^2_I(M')$, which makes $\mathrm{Ass}_R H^2_I(M)$ finite. □

In the rest of this section, the module to which we will map M will be an ideal transform of M with respect to a specially selected subideal of I. We first show that such an ideal transform does in fact have depth 2 on our subideal.

Lemma 1. *Let M be any finitely generated R-module and $x, y \in R$ be non-zerodivisors on M. Then x, y is a possibly improper regular sequence on the ideal transform $D = D_{(x,y)}(M)$.*

Proof. Because x and y are non-zerodivisors on M, the ideal transform $D = M_x \cap M_y = \{u \in M_{xy} | x^N u, y^N u \in M \text{ for some } N\}$. From this reformulation of D it is clear that x is also a non-zerodivisor on D. Suppose that we have $ux = vy$ for some $u, v \in D$. Let $f = \frac{v}{x} = \frac{u}{y}$ in D_{xy}. Then $xf, yf \in D$, so we can find some N for which $x^N xf = x^{N+1} f$, and $y^N yf = y^{N+1} f$ are in M. Thus $f \in D$, so $v = f \cdot x \in xD$ which means x, y is a possibly improper regular sequence on D. □

We control this module D by relating it to the corresponding ideal transform of R.

Lemma 2. *If M is a finitely generated R-module with $Ass_R M \subseteq Ass R$, and $I \subset R$ is any ideal, then $D_I(M)$ is a finitely generated R-module whenever $D_I(R)$ is finitely generated.*

Proof. By [2, Lemma 3.3], $D_I(M)$ is a finitely generated R-module exactly when $D_{IR/P}(R/P)$ is finitely generated for every $P \in \mathrm{Ass}_R M$. But $\mathrm{Ass}_R M \subseteq \mathrm{Ass}_R R$, so $D_I(R)$ finitely generated forces $D_I(M)$ to be finitely generated as well. □

Theorem 1. *Let R be a ring which has $D_{(x,y)}(R)$ a finitely generated module whenever x and y are non-zerodivisors and $ht(x, y) = 2$. Then $Ass_R H^2_I(M)$ is finite whenever $Ass_R M \subseteq Ass_R R$ and $ht(IR/P) \geq 2$ for all $P \in Ass_R M$.*

Proof. We can replace R by $R/\mathrm{Ann}(M)$ and still have $\mathrm{ht}(I) \geq 2$ because $\mathrm{ht}(IR/P) \geq 2$ for any associated prime, P, of M. Pick $x \in I$ so that x is not in any minimal prime of R, so x is a non-zerodivisor on R. Next pick $y \in I - xR$ which is not in any minimal prime of R or xR, which makes y a non-zerodivisor on R, and $\mathrm{ht}(x, y) = 2$.

Let D be the ideal transform $D_{(x,y)}(M)$. By [3, Theorem 2.2.4(i)(c)], we have an exact sequence

$$0 \to H^0_{(x,y)}(M) \to M \to D \to H^1_{(x,y)}(M) \to 0.$$

But $H^0_{(x,y)}(M) = 0$, so we get the short exact sequence which induces

$$\cdots \to H^1_I(D) \to H^1_I(H^1_{(x,y)}(M)) \to H^2_I(M) \to H^2_I(D) \to \cdots$$

Lemma 1 makes x, y a regular sequence on D, so $\text{depth}_I D = 2$. This means our long exact sequence above is actually

$$0 \to H_I^1(H_{(x,y)}^1(M)) \to H_I^2(M) \to H_I^2(D) \to \cdots$$

Since (x, y) is an ideal of height 2 generated by non-zerodivisors, $D_{(x,y)}(R)$ is finitely generated. Lemma 2 then implies D is finitely generated, which means $H_{(x,y)}^1(M) \cong D/M$ is also finitely generated. This means $\text{Ass} H_I^1(H_{(x,y)}^1(M))$ is finite by Proposition 2(c). Because $\text{depth}_I D = 2$, we have that $\text{Ass}_R H_I^2(D)$ is also finite which implies that $\text{Ass}_R H_I^2(M)$ is finite as well. □

Note: One can get also get this result under the stronger condition that R has an S_2-ification for R/P for every prime $P \in \text{Ass}R$, since that forces $D_{(x,y)}(R)$ to be a finitely generated module whenever x and y and non-zerodivisors with $\text{ht}(x, y) = 2$.

We will call a module *skinny* if each of its homomorphic images has only finitely many associated primes. (In particular the module itself must have a finite set of associated primes.)

Lemma 3. *Let R be a Noetherian ring, M any R-module. If the support of M is finite, then M is skinny. In particular this is true of modules with finitely many associated primes all of which are maximal in R.*

Proof. If $M_P = 0$ it is clearly impossible for any quotient, $\bar{M}$, to have $\bar{M}_P \neq 0$. Thus the support of any quotient of M is again finite, which forces its set of associated primes to be finite as well. □

We can use Theorem 1 and the last lemma to show the following:

Theorem 2. *If R has $D_{(x,y)}(R)$ finitely generated whenever x, y are non-zerodivisors with $ht(x, y) = 2$, $dim(R) = 3$ and $ht(IR/P) \geq 2$ for every $P \in Ass_R M$, then $Ass_R H_I^i(M)$ is always finite.*

Proof. By Proposition 1, 2(c), and 3, we only need to show that $\text{Ass}_R H_I^2(M)$ is finite, so if we can reduce to the case where M has pure dimension 3, Theorem 1 applies and we are done.

If $\dim(M) < 3$ then we may work modulo the annihilator of M making $\dim(R) \leq 2$, so by Proposition 3 $H_I^2(M)$ has only finitely many associated primes. Thus we may assume $\dim(M) = 3$.

We next reduce to the case where M has pure dimension. Let $N \subset M$ be the largest submodule of dimension less than 3. Since $\dim(R/P) = 3$ for any $P \in \text{Ass}_R M/N$, we have M/N of pure dimension. In addition since N is M's largest lower dimensional submodule, $\text{Ass}_R M/N \subseteq \text{Ass}_R M$. This means we have $\text{ht}(IR/P) \geq 2$ for every $P \in \text{Ass}_R M/N$, so Theorem 1 tells us $\text{Ass}_R H_I^2(M/N)$ is finite.

The usual short exact sequence $0 \to N \to M \to M/N \to 0$, induces

$$\cdots \to H_I^2(N) \to H_I^2(M) \to H_I^2(M/N) \to H_I^3(N) \to \cdots$$

From this it is immediately clear that if $\dim(N) < 2$ we are done, because then $H_I^2(N) = H_I^3(N) = 0$, which makes $H_I^2(M) \cong H_I^2(M/N)$.

Suppose $\dim(N) = 2$. Here we only have $H_I^3(N) = 0$, which gives us the short exact sequence

$$0 \to \text{im}(H_I^2(N)) \to H_I^2(M) \to H_I^2(M/N) \to 0.$$

It will suffice to show that $H_I^2(N)$ is skinny. We may think over $R/\text{Ann}(N)$, so we may assume that $\dim(R) = 2$. By Proposition 3, we know that $H_I^2(N)$ has only finitely many associated primes, all of which must be of height 2 and thus maximal. Lemma 3 now shows that $H_I^2(N)$ is skinny, so $\text{Ass}_R \text{im}(H_I^2(N))$ is finite. Since $H_I^2(M/N)$ has only finitely many associated primes, $\text{Ass}_R H_I^2(M)$ is therefore finite as well. This means that we have reduced to the case where M has pure dimension 3, so by Theorem 1 we are done. □

2.2. Ideals of height 1

In the previous section, we considered the case where our ideal, I, had height 2 on M so we could use S_2-ification to get to the case where $\text{depth}_I M \geq 2$. In this section, we look at ideals of height 1 on M where such techniques cannot be used. (The height 1 case actually includes the case of height zero ideals, because we can always kill $H_I^0(M) \subseteq M$ so that $\text{depth}_I M \geq 1$.) We show that if R is a local ring of dimension 4 the question of whether $\text{Ass}_R H_I^i(M)$ is finite reduces to the case described below. Some parts of the proof follow the proof of [4, Proposition 2.8]. There is a counterexample in dimension 4 with an ideal of height 1, see [6, Remark 4.2], but there the ring is neither local nor graded suitably for localization. We also discuss the case of a ring of dimension 3 which is not local.

Proposition 5. *Let $S = V[[X_1, \ldots, X_3, Y]]/(f)$ where V is a complete DVR or $V[[X_1, \ldots, X_4, Y]]/(f)$ with V a field, and let f be a monic polynomial in Y with constant term divisible by X_1. If $Ass_S H_{(X_1,Y)}^2(G)$ is finite for any finitely generated faithful module G of pure dimension 4 then $Ass_R H_I^i(M)$ is finite for every four-dimensional local ring (R, m), index i, ideal I, and any finitely generated R-module M.*

Proof. Let (R, m) be local of dimension 4 and pick any ideal $I \subset R$ and finitely generated R-module M. First note that by [4, Proposition 2.8] we are done if $\mathrm{ht}(I) \geq 2$, so we can focus on $\mathrm{ht}(I) \leq 1$. Killing $H^0_I(M) \subseteq M$ does not affect the problem, so we may assume that $\mathrm{depth}_I M \geq 1$. We may also assume that R is complete. By Propositions 1 and 3 and [4, Corollary 2.5], our only possible problem with $\mathrm{Ass}_R H^i_I(M)$ is when $i = 2$.

Choose M to be a minimal counterexample with respect to quotients, i.e., assume that $H^2_I(M/N)$ has only finitely many associated primes for any nonzero submodule $N \subseteq M$. If $\dim(M) \leq 3$ we may work modulo $\mathrm{Ann}(M)$ where we are done by [4, Corollary 2.7], so $\dim(M) = 4$.

Let N be any nonzero submodule of M, so $0 \to N \to M \to M/N \to 0$ induces

$$\cdots \to H^2_I(N) \to H^2_I(M) \to H^2_I(M/N) \to H^3_I(N) \to \cdots$$

which can be broken up as

$$0 \to \mathrm{im}(H^2_I(N)) \to H^2_I(M) \to H^2_I(M/N).$$

When studying $H^2_I(N)$ we may think over $R/\mathrm{Ann}(N)$, so if $\dim(N) < 4$ we have $\dim(R) \leq 3$. But then [4, Corollary 2.5] implies the support of $H^2_I(N)$ is finite, and hence $H^2_I(N)$ is skinny. Since $H^2_I(M/N)$ has only finitely many associated primes by hypothesis, so does $H^2_I(M)$.

We may therefore assume that M has pure dimension 4, and by killing the annihilator of M in R we may also assume M is faithful.

Since $H^i_I(M) = H^i_{\sqrt{I}}(M)$, we may take $I = \bigcap_i P_i$. Let $A = \bigcap_{\mathrm{ht}(P) \leq 1} P$ and $B = \bigcap_{\mathrm{ht}(Q) \geq 2} Q$. This gives us a long exact sequence

$$\cdots \to H^1_{A+B}(M) \to H^2_I(M) \to H^2_A(M) \oplus H^2_B(M) \to H^2_{A+B}(M) \to \cdots$$

The support of $H^i_{A+B}(M)$ is contained in $V(A+B)$. But since $\mathrm{ht}(A+B) \geq 3$, $V(A+B)$ contains only the minimal primes of $(A+B)$ and possibly m. This means the support of $H^i_{A+B}(M)$ is finite and thus it is skinny.

Since $\mathrm{ht}(B) \geq 2$, [4, Proposition 2.8] shows that $\mathrm{Ass}_R H^2_B(M)$ is finite. Thus to show that $H^2_I(M)$ has finitely many associated primes it is enough to see $\mathrm{Ass}_R H^2_A(M)$ is finite, so we may assume that every associated prime of I has height at most 1.

But $\mathrm{depth}_I M \geq 1$ and M is faithful, so I cannot be contained in any minimal prime of R. Thus I is of pure height 1.

Since R is complete, it contains a coefficient ring or field, V. Pick some non-zerodivisor, $x \in I$. Extend $x = x_1$ to a full system of parameters, $\underline{x}$ for R and set $A = V[[\underline{x}]]$. Let $I_0 = \sqrt{(xA)R}$. Clearly $I_0 \subseteq I$, so a primary decomposition of I_0 it will be $(\cap_{i=1}^r P_i) \cap (\cap_{j=1}^s Q_j)$ for some additional primes Q_j. Pick another non-zerodivisor, $y \in I$ with $y \notin Q_j$ for all j, and let

$$J = \sqrt{I_0 + \sqrt{(yA)R}} = \left(\bigcap_{i=1}^r P_i\right) \cap \left(\bigcap_{j=1}^t \widetilde{Q}_j\right).$$

Since all associated primes of J must contain x and y, and the only height one primes containing x are associated to I_0, we must have $\mathrm{ht}(\widetilde{Q}_j) \geq 2$ for all j. By repeating the proof that we have no primes of height 2 in the primary decomposition of $\sqrt{I}$ with $A = I = \bigcap_{i=1}^r P_i$ and $B = \bigcap_{j=1}^t \widetilde{Q}_j$, we can see that if $\mathrm{Ass}_R H^2_I(M)$ is infinite, so is $\mathrm{Ass}_R H^2_J(M)$. Primes in $\mathrm{Ass}_R H^2_{(x,y)}(M)$ lie over primes of $\mathrm{Ass}_{A[x,y]} H^2_{(x,y)}(M)$ and $A \subseteq R$ is module-finite, so if $\mathrm{Ass}_R H^2_{(x,y)}(M)$ is infinite $\mathrm{Ass}_{A[x,y]} H^2_{(x,y)}(M)$ is infinite as well.

We can view $A[x, y] = A[y]$ as a quotient of the ring of formal power series $V[[\underline{X}, Y]]$ by letting the Xs map onto our system of parameters $\underline{x}$ and $Y \mapsto y$. The kernel is clearly a prime of pure height one, and because the formal power series ring is a UFD, that prime is principal so $A[y] \cong V[[\underline{X}, Y]]/(f)$. The Xs form a system of parameters, so some power of Y is dependent on the Xs, which forces f to be monic in Y. Since we cannot have X_1, Y be a regular sequence, we must have some multiple of Y in the ideal generated by X_1. There is only one relation in $A[y]$, so the constant term of our polynomial must be divisible by X_1.

We have now reduced the problem of showing $\mathrm{Ass}_R H^i_I(M)$ is finite for four-dimensional local rings to asking whether $H^2_{(X_1,Y)}(G)$ has finitely many associated primes where G is a faithful module of pure dimension 4 over $V[[\underline{X}, Y]]/(f)$ where f is monic in Y and has constant term divisible by $X = X_1$ as claimed. □

If R is a three dimensional ring that is not local, but which has S_2-ifications for R/P for every $P \in \mathrm{Spec}(R)$, one can use a similar argument to show that the question of when $\mathrm{Ass}_R H^2_I(M)$ is finite reduces to the case where M is a faithful module of pure dimension 3, and I is a radical ideal which is of pure height 1. Unfortunately without the ability to localize our ring and form the completion at its maximal ideal, we are unable to reduce to a quotient of a power series ring as above. Since we know the set of associated primes of any local cohomology module is finite after localizing at any maximal ideal, the problematic case is where we have infinitely many maximal ideals associated to $H^2_I(M)$.

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